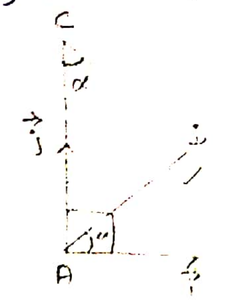
Q12) ABC is a triangle, right angled at A & AD is ⊥r from A to BC. S.T the resultant of the forces acting along AB, AC with magnitudes $1/AB$, $1/AC$ acts along AD & its magnitude is $1/AD$ 47

Soln:

Let $\vec{i}, \vec{j}$ be the unit vectors along AB, AC. Let



$\angle BAD = \alpha$

Now the unit vector along AD is

$$\hat{AD} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$$

$$= \frac{AD}{AB}\vec{i} + \frac{AD}{AC}\vec{j} \rightarrow (1)$$

Now the forces are $\frac{1}{AB}\vec{i}$, $\frac{1}{AC}\vec{j}$ and their resultant is

$$\frac{1}{AB}\vec{i} + \frac{1}{AC}\vec{j} = \frac{1}{AD}\left(\frac{AD}{AB}\vec{i} + \frac{AD}{AC}\vec{j}\right)$$

$$= \frac{1}{AD}\hat{AD} \quad [\text{by } (1)]$$

which is along AD with a magnitude $1/AD$

Q13) The magnitude of the resultant of the forces $\vec{F_1}$ and $\vec{F_2}$ acting on a particle is equal to the magnitude of $\vec{F_1}$. When the first force is doubled S.T the new resultant is ⊥r to $\vec{F_2}$.

Soln:

Since the magnitude of $\vec{F_1} + \vec{F_2}$ is equal to the magnitude of $\vec{F_1}$,

$$|\vec{F_1} + \vec{F_2}| = |\vec{F_1}|$$

$$|\vec{F_1} + \vec{F_2}|^2 = |\vec{F_1}|^2$$

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$$(\vec{F_1}+\vec{F_2})\cdot(\vec{F_1}+\vec{F_2}) = \vec{F_1}\cdot\vec{F_1}$$

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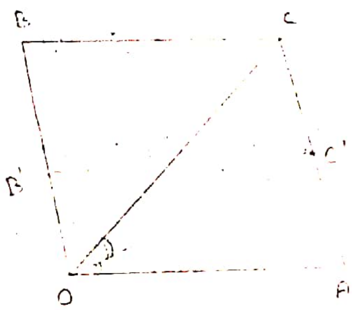
$$\vec{F_1}\cdot\vec{F_1} + 2\vec{F_1}\cdot\vec{F_2} + \vec{F_2}\cdot\vec{F_2} = \vec{F_1}\cdot\vec{F_1}$$

$$(2\vec{F_1}+\vec{F_2})\cdot\vec{F_2} = 0$$

So the resultant of $2\vec{F_1}$ and $\vec{F_2}$ is $\perp$r to $\vec{F_2}$.

Q14) Two forces of equal magnitudes act on a particle and they include an angle θ. If one of them is halved, the angle b/w the other & the orginal resultant is bisected by new resultant. Show that $\theta = 120°$.



Soln :

Let $\vec{OA}$, $\vec{OB}$ be the given forces. Complete the parallelogram OACB. Since OA = OB. OC bisects $\angle AOB$. Let B' be the midpoint of OB. C' is the midpoint of AC. Complete the parallelogram OAC'B'. Since OC' bisects $\angle AOC$,

$$\frac{OA}{OC} = \frac{AC'}{C'C} = 1$$

So, OA = OC, But OA = OB, Hence $\triangle OCA$ is an equilateral triangle & $\angle AOC = 60°$

$$\therefore \angle AOB = 120°$$

Hence Proved.

Q15) Two forces $\vec{P}$ and $\vec{Q}$ which are inclined at an angle α act on a particle. If the sum of their components in certain two perpendicular directions are x and y, show that $\alpha = \cos^{-1}\dfrac{x^2+y^2-P^2-Q^2}{2PQ}$.

Soln:

If R be the resultant of P and Q then, 49

(17)

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \rightarrow (1)$$

Now we know that the algebraic sum of the resolved parts of all the forces about some direction is equal to the resolved part of their resultant,

Hence, R_x = Resolved part of resultant in x direction =

R_y = Resolved part of resultant in y direction =

$$\therefore R = \sqrt{R_x^2 + R_y^2} = \sqrt{x^2+y^2}$$

Using this in (1)

$$x^2 + y^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

$$\cos\alpha = \frac{x^2+y^2-P^2-Q^2}{2PQ}$$

$$\alpha = \cos^{-1}\left(\frac{x^2+y^2-P^2-Q^2}{2PQ}\right)$$

Hence Proved.

Q16) If the resultant of forces 3P, 5P is equal to 7P, find (i) The angle between the forces.
(ii) The angle which the resultant makes with the first force.

Soln:

Given, The resultant of forces 3P, 5P = 7P

(i) Let the angle between the forces = α

$$(7P)^2 = (3P)^2 + (5P)^2 + 2(3P)(5P)\cos\alpha$$

$$49P^2 = 9P^2 + 25P^2 + 30P^2\cos\alpha$$

$$49 = 34 + 30\cos\alpha$$

$$30\cos\alpha = 49 - 34 = 15$$

$$\cos\alpha = 1/2$$

$$\boxed{\alpha = 60^\circ}$$

(ii) Let θ be the angle b/w resultant 7P and the first force 3P.

w.k.T,

$$\tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$= \frac{Q\sin 60^\circ}{P+Q\cos 60^\circ}$$

$$= \frac{Q\cdot\sqrt{3}/2}{P+Q(1/2)}$$

$$= \frac{\sqrt{3}Q}{2P+Q}$$

$$= \frac{\sqrt{3}\cdot 5P}{2(3P)+5P}$$

$$\tan\theta = \frac{5\sqrt{3}}{11}$$

$$\theta = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$

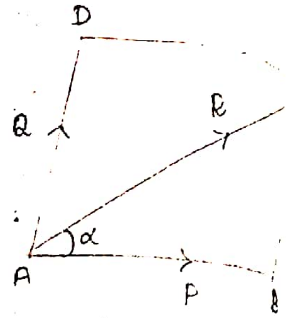
a_{17}) Two forces of magnitudes P and Q act at a point. Their resultant is inclined to the first force at an angle α and has a magnitude R. If the magnitude of the first force is increased by R. S.T the new resultant will make an angle $\alpha/2$ with the first force.

(or)

If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle α with the direction of P, S.T the resultant of forces (P+R) & Q acting at the same angle will make an angle $\alpha/2$ with the direction of P+R.

Soln:

In the figure,

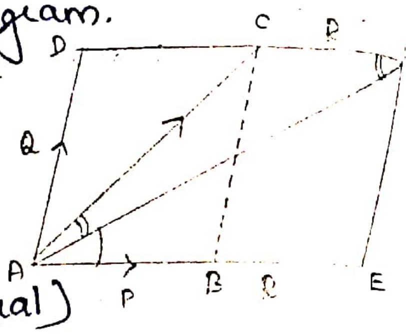


ABCD is a parallelogram

AB = P, AD = Q, AC = R & $\angle BAC = \alpha$.

given,
magnitude of first force P is increased by R

In second figure, AEFD is a parallelogram.



Here R = BE = CF = AC.

$\therefore \Delta AFC$ is isosceles and

$\angle CAF = \angle CFA$ (Base angles are equal)

$= \angle FAE$ (Alternate angle)

Then AF bisects $\angle CAE$

$\Rightarrow \angle FAE = \alpha/2$

Q_{18}) Two equal forces are inclined at an angle 2θ. The magnitude of their resultant is three times the magnitude of the resultant when the forces are inclined at an angle 2ϕ. Showthat $\cos\theta = 3\cos\phi$.

Soln:
Let the two forces be P & Q.

given, P = Q

angle between P & Q = 2θ ie) $\alpha = 2\theta$

$\therefore R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$R^2 = P^2 + P^2 + 2P^2\cos 2\theta \rightarrow (1)$

given, angle inclined at $\alpha = 2\phi$. (52)

$$\therefore R'^2 = P^2 + P^2 + 2P.P\cos 2\phi$$

$$R'^2 = P^2 + P^2 + 2P^2\cos 2\phi \rightarrow (2)$$

given $R = 3R'$

$$\sqrt{P^2+P^2+2P^2\cos 2\theta} = 3\sqrt{P^2+P^2+2P^2\cos 2\phi}$$

$$\Rightarrow P^2+P^2+2P^2\cos 2\theta = 9\left(P^2+P^2+2P^2\cos 2\phi\right)$$

$$2P^2+2P^2\cos 2\theta = 9\left(2P^2+2P^2\cos 2\phi\right)$$

$$2P^2\left[1+\cos 2\theta\right] = 9.2P^2\left(1+\cos 2\phi\right)$$

$$2\cos^2\theta = 9.2\cos^2\phi$$

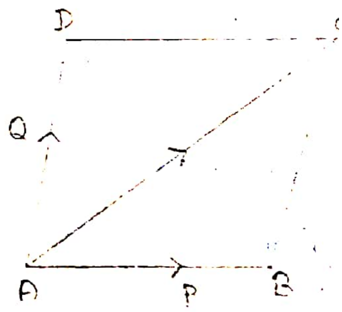
$$\cos^2\theta = 9\cos^2\phi$$

$$\cos\theta = 3\cos\phi$$

Q19) If two forces $\vec{P}, \vec{Q}$ acting at a point is such that their sum & difference are $\perp$r to each other, Show that $P = Q$. Ex:10

Soln:

Let the 2 forces be $\vec{P}$ and $\vec{Q}$. Let the forces P & Q acting at a point A be represented in magnitude and direction by the lines AB and AD.



Complete the parallelogram BADC.

Then $P+Q = \vec{AB}+\vec{AD}$

$= \vec{AC}$ (by parallelogram Law.)

$\therefore \vec{AC}$ is sum of two forces

Now, $P-Q = \vec{AB} - \vec{AD}$

$= \vec{AB} + \vec{DA}$

$= \vec{DA} + \vec{AB}$

$P-Q = \vec{DB}$ (by triangle law).

(14) $\therefore \vec{DB}$ is difference of two forces. (53)

given : $\vec{AC}$ and $\vec{DB}$ are at right angles.

$\therefore$ In parallelogram ABCD, diagonals AC & BD

cut at right angles.

$\Rightarrow \boxed{AB = AD} \Rightarrow P = Q$ in magnitude

Since ABCD must be a rhombus.

$$\therefore P = Q.$$

Q20) The resultant of two forces of magnitude P & Q acting at a point is $\vec{F_1}$. If the second force is replaced with a third force of magnitude R, the new resultant is $\vec{F_2}$. S.T the resultant of $\vec{F_1}$ and the reversed of $\vec{F_2}$ has a magnitude $|Q-R|$

Proof:

Let $\hat{e}_1$ and $\hat{e}_2$ be the unit vectors of the two forces.

given : magnitudes of two forces is $P\hat{e}_1$, $Q\hat{e}_2$

Resultant $= \vec{F_1}$

ie) $P\hat{e}_1 + Q\hat{e}_2 = \vec{F_1}$

given: Second force is replaced with a third force of magnitude R. and new resultant $= \vec{F_2}$

ie) $P\hat{e}_1 + R\hat{e}_2 = \vec{F_2}$

To Show: resultant of $\vec{F_1}$ and the reversed resulta of $\vec{F_2}$ is $|Q-R|$.

ie) $|\vec{F_1} - \vec{F_2}| = |Q-R|$.

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Now, $|\vec{F_1} - \vec{F_2}| = |P\hat{e}_1 + Q\hat{e}_2 - (P\hat{e}_1 + R\hat{e}_2)|$

$= |(Q-R)\hat{e}_2|$

$= |Q-R|\,|\hat{e}_2|$

$= |Q-R|$ Since $|\hat{e}_2| = 1$

$\therefore |\vec{F_1} - \vec{F_2}| = |Q-R|$.

Q21) Two forces $\vec{P}$ & $\vec{Q}$ acting at a point have the resultant $\vec{R}$. when the first force is doubled, the new resultant bisects the angle b/w $\vec{R}$ & $\vec{P}$. Show that $|\vec{P}| = |\vec{R}|$.

Proof:

given, $\vec{R} = \vec{P} + \vec{Q} \rightarrow (1)$

and $2P, Q \rightarrow$ new resultant bisects the angle b/w $\vec{R}$ & $\vec{P}$

$\therefore 2\vec{P} + \vec{Q} = \vec{P} + (\vec{P} + \vec{Q})$

$= \vec{P} + \vec{R}$ by (1)

$\therefore 2\vec{P} + \vec{Q}$ is resultant of P and R. This bisects the angle b/w R and P.

$\therefore |\vec{P}| = |\vec{R}|$.

Note:

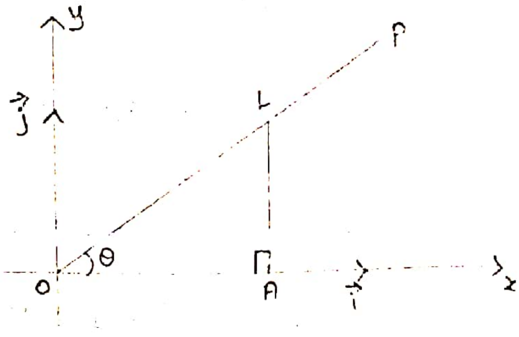
Let the straight line OP makes angle θ with the x-axis and let the unit vector be $\hat{n}$.

Now $OL = 1$ & $\vec{n} = OL$

$\vec{n} = \vec{OA} + \vec{AL}$

$= OA\vec{i} + AL\vec{j}$

$\vec{n} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$



$OL = 1$

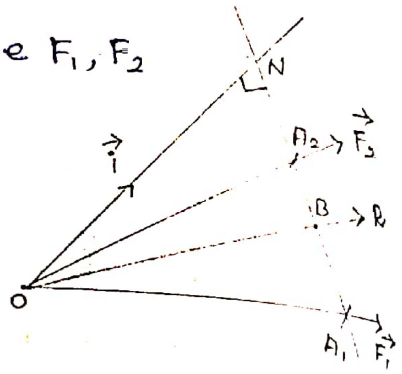
$\frac{OA}{OL} = \cos\theta \Rightarrow OA = \cos\theta$

$\frac{AL}{OL} = \sin\theta \Rightarrow AL = \sin\theta$

Q22) The resultant of the forces F_1, F_2 acting at O is R. If any transversal meets, the lines of action of F_1, F_2, R at A_1, A_2, B prove that, $\frac{F_1}{OA_1} + \frac{F_2}{OA_2} = \frac{R}{OB}$.

Proof:

Let the two forces acting at O be F_1, F_2



Resultant = R:

Draw ON $\perp^r$ to transversal.

Let $\vec{i} \to$ unit vector along ON.

$\vec{F_1}, \vec{F_2} \to$ two forces.

$\vec{R} \to$ the resultant

ie) $\vec{R} = \vec{F_1} + \vec{F_2} \to (1)$

Angle, between $\vec{i}$ & the forces $\vec{F_1}$ is $\angle A_1ON$.

" " " " $\vec{F_2}$ is $\angle A_2ON$

" " " " $\vec{R}$ is $\angle BON$

multiply (1) scalarly by $\vec{i}$, we get

$$\vec{F_1}\cdot\vec{i} + \vec{F_2}\cdot\vec{i} = \vec{R}\cdot\vec{i}$$

$$\therefore F_1 \cdot 1 \cdot \cos(A_1ON) + F_2 \cdot 1 \cdot \cos A_2ON = R \cdot 1 \cdot \cos BON$$

ie) $$F_1 \frac{ON}{OA_1} + F_2 \frac{ON}{OA_2} = R \frac{ON}{OB}.$$

$$\Rightarrow \frac{F_1}{OA_1} + \frac{F_2}{OA_2} = \frac{R}{OB}.$$

Hence Proved.

Remark:

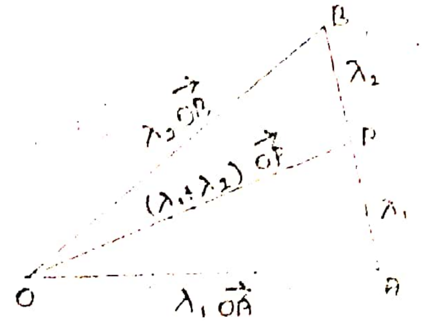
The above result can be extended to n forces $F_1, F_2, \dots, F_n$ acting at O.

Then we have,

$$\lambda_1 \frac{F_1}{OA_1} + \lambda_2 \frac{F_2}{OA_2} + \dots + \lambda_n \frac{F_n}{OA_n} = \frac{R}{OB}$$

where, $\lambda_r = 1$ or -1 according as $A_i O H$ is positive (or) Negative.

a23) S.T the resultant of the forces $\lambda_1 \vec{OA}$, $\lambda_2 \vec{OB}$ is $(\lambda_1 + \lambda_2)\vec{OP}$, where P divides AB in the ratio $\lambda_2 : \lambda_1$.



Proof: [$(\lambda-\mu)$ theorem]

$$\vec{OA} = \vec{OP} + \vec{PA}$$

$$\therefore \lambda_1 \vec{OA} = \lambda_1 \vec{OP} + \lambda_1 \vec{PA} \rightarrow (1)$$

$|||^{ly}$ $\vec{OB} = \vec{OP} + \vec{PB}$

$$\therefore \lambda_2 \vec{OB} = \lambda_2 \vec{OP} + \lambda_2 \vec{PB} \rightarrow (2)$$

$(1)+(2) \Rightarrow$

$$\lambda_1 \vec{OA} + \lambda_2 \vec{OB} = (\lambda_1 + \lambda_2)\vec{OP} + \lambda_1 \vec{PA} + \lambda_2 \vec{PB}$$

We know that,

$$\lambda_1 AP = \lambda_2 PB$$

$$(2) \Rightarrow \lambda_1 \vec{OA} + \lambda_2 \vec{OB} = (\lambda_1 + \lambda_2)\vec{OP}$$

Hence proved.

Corollary:

$$\lambda_1 = \lambda_2 = 1$$

$\Rightarrow$ P is mid point of AB

$$\Rightarrow \vec{OA} + \vec{OB} = 2\vec{OP}.$$

Resultant of three forces related to a triangle acting at a point:

Consider, in particular forces pertaining to a triangle and acting at a point. Here the following formulae will be used:

1. $\overline{AB} + \overline{BC} + \overline{CA} = 0$
2. If M is the midpoint of AB, then $\overline{OM} = \frac{1}{2}(\overline{OA} + \overline{OB})$
3. If G is the Centroid of the ΔABC, then

$$\overline{OG} = \frac{1}{3}(\overline{OA} + \overline{OB} + \overline{OC})$$

$$3\overline{OG} = \overline{OA} + \overline{OB} + \overline{OC}.$$

Def: Equilibrium:

When the resultant of the forces acting at a point is zero, then the forces are said to be in equilibrium.

Vector addition formula:

1. Position Vector of a point which divides AB in ratio is given by

$$\vec{OM} = \frac{n\vec{OA} + m\vec{OB}}{m+n}$$

2. Position Vector of Midpoints of AB is

$$= \frac{m\vec{OA} + m\vec{OB}}{2m}$$

$$\vec{OM} = \frac{\vec{OA} + \vec{OB}}{2}$$

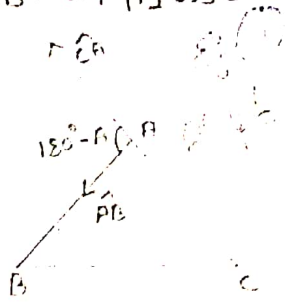
3. If the position vector of the vertex of a triangle are $\vec{a}, \vec{b}$ & $\vec{c}$ then the P.V of the Centroid is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

where median is the line joining the midpoint of side and is opposite. $\vec{OG} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

Problems:

B₁) Forces of magnitudes F_1, F_2, F_3 act on a particle. If their directions are parallel to $\vec{BC}$, $\vec{CA}$, $\vec{AB}$ where ABC is a triangle, show that the magnitude of their resultant is $\sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3\cos A - 2F_3F_1\cos B - 2F_1F_2\cos C}$

Soln:



The given forces are $F_1\hat{BC}$, $F_2\hat{CA}$, $F_3\hat{AB}$ where, $\hat{BC}$, $\hat{CA}$, $\hat{AB}$ are the unit vectors parallel to $\vec{BC}$, $\vec{CA}$, $\vec{AB}$. If the magnitude of their resultant is F, then we have

$$F^2 = (F_1\hat{BC} + F_2\hat{CA} + F_3\hat{AB})\cdot(F_1\hat{BC} + F_2\hat{CA} + F_3\hat{AB})$$

$$= F_1^2 + F_2^2 + F_3^2 + 2F_2F_3\,\hat{CA}\cdot\hat{AB} + 2F_3F_1\,\hat{AB}\cdot\hat{BC} + 2F_1F_2\,\hat{BC}\cdot\hat{CA}.$$

$$= F_1^2 + F_2^2 + F_3^2 - 2F_2F_3\cos A - 2F_3F_1\cos B - 2F_1F_2\cos C$$

because $\hat{CA}\cdot\hat{AB} = \cos(180^\circ - A) = -\cos A$ etc.

$$F = \sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3\cos A - 2F_3F_1\cos B - 2F_1F_2\cos C}$$

Hence proved.

B₂) Three forces of equal magnitudes P act on a particle. If their directions are parallel to the sides BC, CA, AB of a triangle ABC, show that the magnitude of their resultant is $P\sqrt{(3 - 2\cos A - 2\cos B - 2\cos C)}$

Soln:

Let the given forces are $P\hat{BC}$, $P\hat{CA}$, $P\hat{AB}$ where $\hat{BC}$, $\hat{CA}$, $\hat{AB}$ are the unit vector parallel to $\vec{BC}$, $\vec{CA}$, $\vec{AB}$.

Let 'p' be the magnitude of the resultant force.

$\vec{P} = P\hat{BC} + P\hat{CA} + P\hat{AB}$

$P^2 = (P\hat{BC} + P\hat{CA} + P\hat{AB}) \cdot (P\hat{BC} + P\hat{CA} + P\hat{AB})$

$= P^2 + P^2 + P^2 + P^2\,\hat{BC}\cdot\hat{CA} + P^2\,\hat{BC}\cdot\hat{AB} + P^2\,\hat{CA}\cdot\hat{BC} + P^2\,\hat{CA}\cdot\hat{AB}$
$+ P^2\,\hat{AB}\cdot\hat{BC} + P^2\,\hat{AB}\cdot\hat{CA}$

$= P^2 + P^2 + P^2 + 2P^2\,\hat{BC}\cdot\hat{CA} + 2P^2\,\hat{CA}\cdot\hat{AB} + 2P^2\,\hat{AB}\cdot\hat{BC}$

$= 3P^2 + 2P^2(-\cos C) + 2P^2(-\cos A) + 2P^2(-\cos B)$

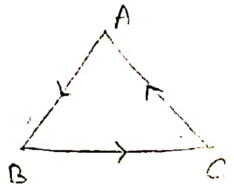
$P = P\sqrt{3 - 2\cos A - 2\cos B - 2\cos C}$

B3) Three forces acting at a point are parallel to the sides of a triangle ABC, taken in order, and in magnitude they are proportional to the cosines of the opposite angles. Show that the magnitude of their resultant is proportional to $\sqrt{(1 - 8\cos A\cos B\cos C)}$.

Soln:

Let be the given three forces $k\cos A\,\hat{BC}$, $k\cos B\,\hat{CA}$, $k\cos C\,\hat{AB}$.

Let F be the resultant.



$\vec{F} = (k\cos A\,\hat{BC} + k\cos B\,\hat{CA} + k\cos C\,\hat{AB})$

$F^2 = (k\cos A\,\hat{BC} + k\cos B\,\hat{CA} + k\cos C\,\hat{AB}) \cdot (k\cos A\,\hat{BC} + k\cos B\,\hat{CA}$
$+ k\cos C\,\hat{AB})$

$= k^2\cos^2 A + k^2\cos^2 B + k^2\cos^2 C + k^2\cos A\cos B\,\hat{BC}\cdot\hat{CA} +$
$k^2\cos A\cos C\,\hat{BC}\cdot\hat{AB} + k^2\cos A\cos B\,\hat{BC}\cdot\hat{CA} +$
$k^2\cos B\cos C\,\hat{AB}\cdot\hat{CA} + k^2\cos A\cos C\,\hat{AB}\cdot\hat{BC} +$
$k^2\cos B\cos C\,\hat{AB}\cdot\hat{CA}.$

$$= k^2\Big[\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\ \hat{BC}\cdot\hat{CA} + 2\cos A\cos C\ \hat{BC}\cdot\hat{AB} + 2\cos B\cos C\ \hat{AB}\cdot\hat{CA}\Big]$$

$$= k^2\Big[\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C - 2\cos A\cos C\cos B - 2\cos A\cos B\cos C\Big]$$

$$= k^2\Big[\underbrace{\cos^2 A + \cos^2 B + \cos^2 C}_{= 1 - 2\cos A\cos B\cos C} - 6\cos A\cos B\cos C\Big]$$

$$F = k\sqrt{1 - 8\cos A\cos B\cos C}$$

$$F \propto \sqrt{1 - 8\cos A\cos B\cos C}$$

B4) The sides BC, CA, AB of a ΔABC are bisected in D, E, F Show that the forces represented by DA, EB, FC are in equilibrium.

Eg. proof

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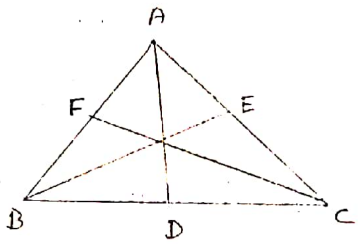
Soln:



The forces act through the Centroid, Since D is the midpoint of BC,

$$\vec{AD} = \frac{1}{2}(\vec{AB} + \vec{AC}) = \frac{1}{2}(\vec{AB} - \vec{CA})$$

But $\vec{DA} = -\vec{AD}$

Hence

$$\vec{DA} = -\tfrac{1}{2}(\vec{AB} - \vec{CA})$$

$$\vec{EB} = -\tfrac{1}{2}(\vec{BC} - \vec{AB})$$

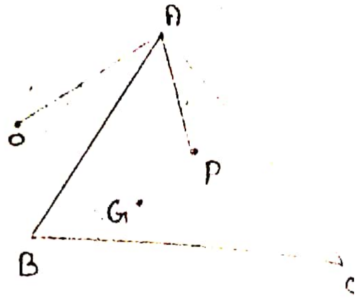
$$\vec{FC} = -\tfrac{1}{2}(\vec{CA} - \vec{BC})$$

Adding these three, we get the resultant as $\vec{0}$. So, the forces are in equilibrium.

(61)

B5) ABC is a triangle. G is its Centroid and P is any point in the plane of the triangle. S.T the resultant of forces represented by $\vec{PA}, \vec{PB}, \vec{PC}$ is $3\vec{PG}$ and find the position of P, if the three forces are in equilibrium.

Soln:



Let O be any base point. Then it is evident that,

$$\vec{PA} = \vec{PG} + \vec{GO} + \vec{OA}$$

$$\therefore \Sigma \vec{PA} = 3\vec{PG} + 3\vec{GO} + (\vec{OA} + \vec{OB} + \vec{OC})$$

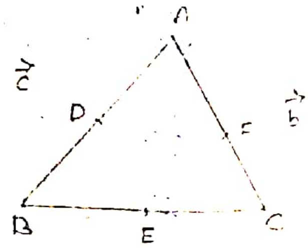
$$= 3\vec{PG} - 3\vec{OG} + 3\vec{OG}$$

$$\Sigma \vec{PA} = 3\vec{PG}$$

This becomes Zero and the forces are in equilibrium when P Coincides with the Centroid G.

B6) D, E, F are midpoints of the sides of a triangle ABC. If O is any point in space, show that the forces $\vec{OA}, \vec{OB}, \vec{OC}, \vec{DO}, \vec{EO}, \vec{FO}$ are in equilibrium.

Soln:



Let $\vec{a}, \vec{b}, \vec{c}$ be the vectors represents $\vec{AB}, \vec{BC}, \vec{CA}$

Let O be a point on the space & D, E, F are midpoints of $\vec{AB}, \vec{BC}, \vec{CA}$ respectively

$$\vec{OD} = \frac{1}{2}(\vec{OA} + \vec{OB}) \Rightarrow \vec{DO} = -\frac{1}{2}(\vec{OA} + \vec{OB}) \rightarrow (1)$$

$$\vec{OE} = \frac{1}{2}(\vec{OB} + \vec{OC}) \Rightarrow \vec{EO} = -\frac{1}{2}(\vec{OB} + \vec{OC}) \rightarrow (2)$$

$$\vec{FO} = -\frac{1}{2}(\vec{OA} + \vec{OC}) \rightarrow (3)$$

$$\vec{OA}+\vec{OB}+\vec{OC}+\vec{DO}+\vec{EO}+\vec{FO} = \vec{OA}+\vec{OB}+\vec{OC} - \frac{1}{2}(\vec{OA}+\vec{OB}+\vec{OB}+\vec{OC}+\vec{OA}+\vec{OC})$$

$$= \vec{OA}+\vec{OB}+\vec{OC} - (\vec{OA}+\vec{OB}+\vec{OC})$$ 62

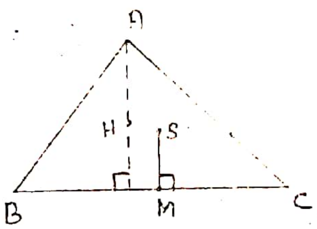
$$= 0$$

Hence proved.

31) S and H are the circumcentre and orthocentre of a triangle ABC. Show that

(i) the resultant of the forces $\vec{SA}, \vec{SB}, \vec{SC}$ acting at S is $\vec{SH}$.

(ii) The resultant of the forces $\vec{HA}, \vec{HB}, \vec{HC}$ acting at H is $2\vec{HS}$



Soln:

Let M be the midpoint of BC. Then M divides BC in the ratio 1:1. So

$$\vec{SM} = \frac{\vec{SB}+\vec{SC}}{2}$$

$$\vec{SB}+\vec{SC} = 2\vec{SM} \rightarrow (1)$$

IIIly $\vec{HB}+\vec{HC} = 2\vec{HM} \rightarrow (2)$

But we know, from geometry, that AH is parallel to SM and AH = 2SM. So

$$\vec{AH} = 2\vec{SM} \rightarrow (3)$$

(i) $\vec{SA}+\vec{SB}+\vec{SC} = \vec{SA}+2\vec{SM}$ by (1)

$= \vec{SA}+\vec{AH}$ by (3)

$= \vec{SH}$

(ii) $\vec{HA}+\vec{HB}+\vec{HC} = \vec{HA}+2\vec{HM}$ by (2)

$= 2\vec{MS}+2\vec{HM}$ by (3)

$= 2(\vec{MS}+\vec{HM})$.

$= 2(\vec{HM}+\vec{MS})$

$= 2\vec{HS}$

Hence proved

Resultant of several forces acting on a particle. 63

Book Work:

To find the resultant of Coplanar forces using their Components.

Let us find the resultant of the forces $\vec{F_1}, \vec{F_2}, \ldots, \vec{F_n}$. Now the resultant is

$$\vec{F_1} + \vec{F_2} + \cdots + \vec{F_n} = x\vec{i} + y\vec{j} \text{ say}$$

Multiplying this Scalarly by $\vec{i}$

$$\vec{F_1}\cdot\vec{i} + \vec{F_2}\cdot\vec{i} + \cdots + \vec{F_n}\cdot\vec{i} = x\vec{i}\cdot\vec{i} = x$$

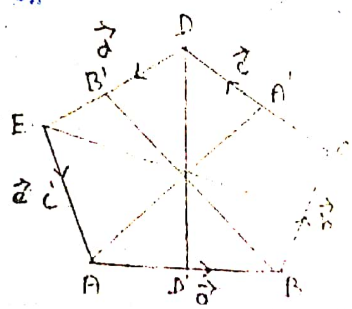
Now $\vec{F_1}\cdot\vec{i}$ is the component of $\vec{F_1}$ in the $\vec{i}$ direction. So x is the sum of the Components of the forces in the $\vec{i}$ direction. Similarly y is the sum of the Components of the forces in the $\vec{j}$ direction. Now the magnitude of the resultant is

$$|x\vec{i} + y\vec{j}| = \sqrt{x^2+y^2}$$

and the angle between the resultant and the $\vec{i}$ direction is $\tan^{-1}\frac{y}{x}$.

Problems:

(1) Five forces acting at a point are represented in magnitude and direction by the lines joining the Vertices of any pentagon to the midpoints of their opposite sides. Show that they are in equilibrium.



Soln: Let ABCDE be the pentagon & A', B', C', D', E' be the midpoints of

Sides opposite to A, B, C, D, E. Let $AB = a$, $BC = b$ ..... $EA = e$

Then, $\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e} = \vec{0}$.

Now, Considering the force $\vec{AA'}$, we have

$$\vec{AA'} = \vec{AB} + \vec{BC} + \vec{CA'}$$

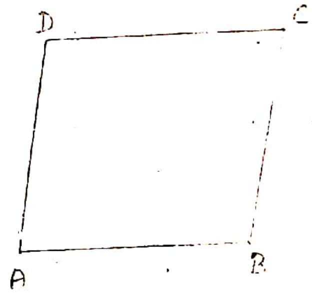
$$= \vec{a} + \vec{b} + \frac{\vec{c}}{2}$$

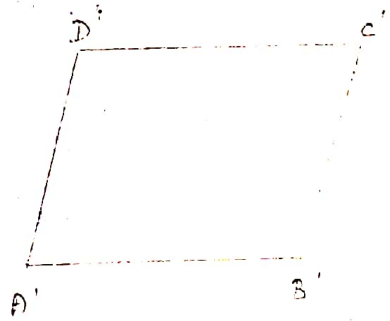
$$\Sigma \vec{AA'} = \frac{5}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e})$$

$$= \vec{0}$$

Hence Proved.

C$_2$) Show that the forces $\vec{AA'}$, $\vec{BB'}$, $\vec{CC'}$, $\vec{D'D}$ acting at a point, where ABCD, A'B'C'D' are two parallelograms in space are in equilibrium.





Soln:

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{a'}, \vec{b'}, \vec{c'}, \vec{d'}$ be the position Vector. A, B, C, D, A', B', C', D' respectively

$$\vec{AB} = \vec{DC}$$

$$\vec{OB} - \vec{OA} = \vec{OC} - \vec{OD}$$

$$\vec{b} - \vec{a} = \vec{c} - \vec{d}$$

$$-\vec{a} + \vec{b} - \vec{c} + \vec{d} = 0 \rightarrow (1)$$

III^{ly} for the second figure of parallelogram, we have,

$$-\vec{a'} + \vec{b'} - \vec{c'} + \vec{d'} = 0 \rightarrow (2)$$

$$\vec{AA'} = \vec{OA'} - \vec{OA} = \vec{a'} - \vec{a}$$

$$\vec{B'B} = \vec{OB} - \vec{OB'} = \vec{b} - \vec{b'}$$

$\vec{CC'} = \vec{OC'} - \vec{OC} = \vec{c'} - \vec{c}$

$\vec{D'D} = \vec{OD} - \vec{OD'} = \vec{d} - \vec{d'}$

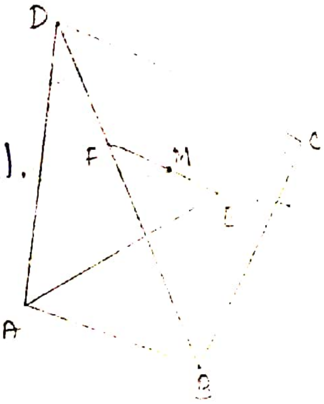
$$\vec{AA'} + \vec{B'B} + \vec{CC'} + \vec{D'D} = \left[-\vec{a} + \vec{b} - \vec{c} + \vec{d}\right] - \left[-\vec{a'} + \vec{b'} - \vec{c'} + \vec{d'}\right]$$

$$= 0 - 0$$

$$= 0 \quad \text{from } ① \,\&\, ②$$

(3) ABCD is a quadrilateral Find a point 'O' such that the forces $\vec{OA}, \vec{OB}, \vec{OC}, \vec{OD}$ acting at O may be in equilibrium.

Soln:



Let A, B, C, D be the quadrilateral.

Let E, F be the midpoint of AC & BD respectively. and M be the midpoint EF.

$$\vec{OE} = \frac{\vec{OA} + \vec{OC}}{2}$$

$$2\vec{OE} = \vec{OA} + \vec{OC} \longrightarrow ①$$

$$\vec{OF} = \frac{\vec{OB} + \vec{OD}}{2}$$

$$2\vec{OF} = \vec{OB} + \vec{OD} \longrightarrow ②$$

① + ②

$$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2(\vec{OE} + \vec{OF})$$

$$= 2(2\vec{OM})$$

$$= 4\vec{OM}$$

If the point 'O' Conicides with the point 'M'. then the forces are in equilibrium.

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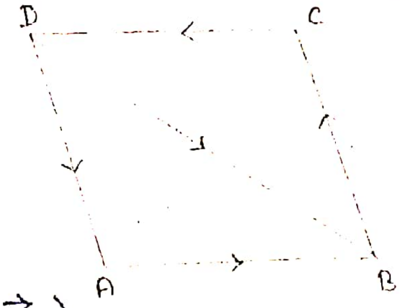
(4) Forces acting at a point are represented in magnitude and direction by $\vec{AB}$, $2\vec{BC}$, $2\vec{CD}$, $\vec{DA}$, $\vec{DB}$ where ABCD is a square. Show that the forces are in equilibrium.



Soln:

The resultant of the given forces

$$= \vec{AB} + 2\vec{BC} + 2\vec{CD} + \vec{DA} + \vec{DB}$$

$$= (\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA}) + (\vec{BC} + \vec{CD} + \vec{DB})$$

The forces in the first brackets are in equilibrium by the polygon of forces (here the square ABCD) and the forces in the second brackets are in equilibrium by the triangle of forces (here the triangle BCD).

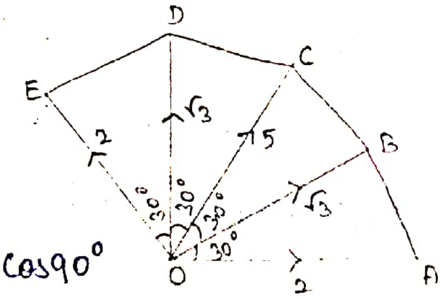
$$\therefore \vec{AB} + 2\vec{BC} + 2\vec{CD} + \vec{DA} + \vec{DB} = 0$$

ie) The given set of forces are in equilibrium.

(5) Forces of magnitudes 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 respectively act at one of the angular points of a regular hexagon towards the other five points in order. Show that their resultant is of magnitude 10 and makes an angle of 60° with the first force.

Soln:

Resolving the forces along OA and perpendicular to OA.



$$R_x = OA\cos 0° + OB\cos 30° + OC\cos 60° + OD\cos 90° + OE\cos 120°$$

$$= 2 + \sqrt{3}\cos 30° + 5\cos 60° + \sqrt{3}\cos 90° + 2\cos 120°$$

$$= 2 + \sqrt{3}\cdot\frac{\sqrt{3}}{2} + 5\cdot\frac{1}{2} + \sqrt{3}(0) + 2\left(-\frac{1}{2}\right)$$